

A REPLICA-COUPLING APPROACH TO DISORDERED PINNING MODELS

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ABSTRACT. We consider a renewal process $\tau = \{\tau_0, \tau_1, \dots\}$ on the integers, where the law of $\tau_i - \tau_{i-1}$ has a power-like tail $\mathbf{P}(\tau_i - \tau_{i-1} = n) = n^{-(\alpha+1)}L(n)$ with $\alpha \geq 0$ and $L(\cdot)$ slowly varying. We then assign a random, n -dependent reward/penalty to the occurrence of the event that the site n belongs to τ . In such generality this class of problems includes, among others, $(1+d)$ -dimensional models of pinning of directed polymers on a one-dimensional random defect, $(1+1)$ -dimensional models of wetting of disordered substrates, and the Poland-Scheraga model of DNA denaturation. By varying the average of the reward, the system undergoes a transition from a *localized phase* where τ occupies a finite fraction of \mathbb{N} to a *delocalized phase* where the density of τ vanishes. In absence of disorder (i.e., if the reward is independent of n), the transition is of first order for $\alpha > 1$ and of higher order for $\alpha < 1$. Moreover, for α ranging from 1 to 0, the transition ranges from first to infinite order. Presence of even an arbitrarily small (but extensive) amount of disorder is known to modify the order of transition as soon as $\alpha > 1/2$ [11]. In physical terms, disorder is relevant in this situation, in agreement with the heuristic Harris criterion. On the other hand, for $0 < \alpha < 1/2$ it has been proven recently by K. Alexander [2] that, if disorder is sufficiently weak, critical exponents are not modified by randomness: disorder is irrelevant. In this work, generalizing techniques which in the framework of spin glasses are known as *replica coupling* and *interpolation*, we give a new, simpler proof of the main results of [2]. Moreover, we (partially) justify a small-disorder expansion worked out in [9] for $\alpha < 1/2$, showing that it provides a free energy upper bound which improves the annealed one.

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1. INTRODUCTION

Consider a (recurrent or transient) Markov chain $\{S_n\}_{n \geq 0}$ started from a particular point, call it 0 by convention, of the state space Σ . Assume that the distribution of the inter-arrival times to the state 0 has a power-like tail: if $\tau := \{n \geq 0 : S_n = 0\}$, we require $\mathbf{P}(\tau_i - \tau_{i-1} = n) \simeq n^{-\alpha-1}$ for n large (see Eq. (2.1) below for precise definitions and conditions). This is true, for instance, if S is the simple random walk (SRW) in $\Sigma = \mathbb{Z}^d$, in which case $\alpha = 1/2$ for $d = 1$ and $\alpha = d/2 - 1$ for $d \geq 2$. One may naturally think of $\{(n, S_n)\}_{n \geq 0}$ as a directed polymer configuration in $\Sigma \times \mathbb{N}$. We want to model the situation where the polymer interacts with the one-dimensional defect line $\{0\} \times \mathbb{N}$. To this purpose, we introduce the Hamiltonian

$$\mathcal{H}_N(S) = - \sum_{n=1}^N \varepsilon_n \mathbf{1}_{S_n=0} \quad (1.1)$$

which gives a reward (if $\varepsilon_n > 0$) or a penalty (if $\varepsilon_n < 0$) to the occurrence of a polymer-line contact at step n . Typically, we have in mind the situation where $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is a sequence of IID (possibly degenerate) random variables. Let h and β^2 be the average and variance of ε_n , respectively. Varying h at β fixed, the system undergoes a phase transition: for $h > h_c(\beta)$ the Boltzmann average of the contact fraction $\ell_N := |\{1 \leq n \leq N : S_n = 0\}|/N$ converges almost surely to a positive constant, call it $\ell(\beta, h)$, for $N \rightarrow \infty$ (localized phase), while for $h < h_c(\beta)$ it converges to zero (delocalized phase). Models of this kind are employed in the physics literature to describe, for instance, the interaction of $(1+1)$ -dimensional interfaces with disordered walls [6], of flux lines with columnar defects in type-II superconductors [17], and the DNA denaturation transition in the Poland-Scheraga approximation [5].

In absence of disorder ($\beta = 0$) it is known that the transition is of first order ($\ell(0, h)$ has a discontinuity at $h_c(0)$) if $\alpha > 1$, while for $0 \leq \alpha < 1$ the transition is continuous: in particular, $\ell(0, h)$ vanishes like $(h - h_c(0))^{(1-\alpha)/\alpha}$ for $h \searrow h_c(0)$ if $0 < \alpha < 1$ and faster than any power of $(h - h_c(0))$ if $\alpha = 0$. For $\alpha = 1$, finally, transition can be either continuous or discontinuous, depending on the slowly varying function $L(\cdot)$ in (2.1). An interesting question concerns the effect of disorder on the nature of the transition. In terms of the non-rigorous Harris criterion, disorder is believed to be irrelevant for $\alpha < 1/2$ and relevant for $\alpha > 1/2$, where “relevance” refers to the property of changing the critical exponents. The question of disorder relevance in the (so called “marginal”) case $\alpha = 1/2$ is not settled yet, even on heuristic grounds. Recently, rigorous methods have allowed to put this belief on firmer ground. In particular, in Refs. [11]-[12] it was proved that, for every $\beta > 0$, $\alpha \geq 0$ and h sufficiently close to (but larger than) $h_c(\beta)$, one has $\ell(\beta, h) \leq (1+\alpha)c(\beta)(h - h_c(\beta))$, for some $c(\beta) < \infty$. This result, compared with the critical behavior mentioned above of the non-random model, proves relevance of disorder for $\alpha > 1/2$, since $1 > (1-\alpha)/\alpha$. On the other hand, in a recent remarkable work K. Alexander showed [2] that the opposite is true for $0 < \alpha < 1/2$: if disorder is sufficiently weak, $\ell(\beta, h)$ vanishes like $(h - h_c(\beta))^{(1-\alpha)/\alpha}$ as in the homogeneous model. Moreover, the critical point $h_c(\beta)$ coincides, always for β small and $0 < \alpha < 1/2$, with the critical point $h_c^a(\beta)$ of the corresponding *annealed model* (cf. Section 2). Always in [2], for $1/2 \leq \alpha < 1$ it was proven that the ratio $h_c(\beta)/h_c^a(\beta)$ converges to 1 for $\beta \searrow 0$. This, on the other hand, is expected to be false for $\alpha > 1$.

The purpose of this work is twofold. Firstly, we present a method which allows to re-obtain the main results of [2] in a simpler way. Secondly, we show that the well known inequality between quenched and annealed free energies is strict as soon as the annealed model is localized and $\beta > 0$. Moreover, we prove that a small-disorder expansion for the quenched free energy, worked out in [9] for $0 \leq \alpha < 1/2$, provides at least a free energy upper bound.

As far as the first point is concerned, our strategy is a generalization of techniques which in the domain of mean field spin glasses are known as *replica coupling* [18] [15] and *interpolation*. These methods had a remarkable impact on the understanding of spin glasses in recent years (see, e.g., [16], [14], [1], [19]). In particular the “quadratic replica coupling” method, introduced in [15], gives a very efficient control of the Sherrington-Kirkpatrick model at high temperature (β small), i.e., for weak disorder, which is the same situation we are after here. Our method is not unrelated to that of [2]: the two share the idea that the basic object to look at is the law of the intersection of two independent copies of the renewal τ . However, our strategy allows to bypass the need of performing refined second-moment computations on a suitably truncated partition function as in [2] and gives, in the case of Gaussian disorder, particularly neat proofs.

In the rest of the paper, we will forget the polymer-like interpretation and the Markov chain structure, and define the model directly starting from the process τ of the “returns to 0”.

2. MODEL AND RESULTS

Consider a *recurrent* renewal sequence $0 = \tau_0 < \tau_1 < \dots$ where $\{\tau_i - \tau_{i-1}\}_{i \geq 0}$ are integer-valued IID random variables with law

$$K(n) := \mathbf{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}} \quad \forall n \in \mathbb{N}. \quad (2.1)$$

We assume that the function $L(\cdot)$ is slowly varying at infinity [4], $\alpha \geq 0$ and $\sum_{n \in \mathbb{N}} K(n) = 1$. Recall that a slowly varying function $L(\cdot)$ is a positive function $(0, \infty) \ni x \rightarrow L(x) \in (0, \infty)$ such that, for every $r > 0$,

$$\lim_{x \rightarrow \infty} \frac{L(rx)}{L(x)} = 1. \quad (2.2)$$

We denote by \mathbf{E} the expectation on $\tau := \{\tau_i\}_{i \geq 0}$ and we put for notational simplicity $\delta_n := \mathbf{1}_{n \in \tau}$, where $\mathbf{1}_A$ is the indicator of the event A .

We define, for $\beta \geq 0$ and $h \in \mathbb{R}$, the quenched free energy as

$$F(\beta, h) = \lim_{N \rightarrow \infty} F_N(\beta, h) := \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \log \mathbf{E} \left(e^{\sum_{n=1}^N (\beta \omega_n + h) \delta_n} \delta_N \right) \quad (2.3)$$

where $\{\omega_n\}_{n \in \mathbb{N}}$ are IID centered random variables with finite second moment, law denoted by \mathbb{P} and corresponding expectation \mathbb{E} , and normalized so that $\mathbb{E} \omega_1^2 = 1$. In this work, we restrict to the case where disorder has a Gaussian distribution: $\omega_1 \stackrel{d}{=} \mathcal{N}(0, 1)$. Some degree of generalization is possible: for instance, results and proofs can be extended to the situation where ω_n are IID bounded random variables.

The existence of the $N \rightarrow \infty$ limit in (2.3) is well known, see for instance [9, Section 4.2]. The limit actually exists, and is almost-surely equal to $F(\beta, h)$, even omitting the disorder average \mathbb{E} in (2.3). We point out that, by superadditivity, for every $N \in \mathbb{N}$

$$F_N(\beta, h) \leq F(\beta, h) \quad (2.4)$$

and that, from Jensen's inequality,

$$F_N(\beta, h) \leq F_N^a(\beta, h) := \frac{1}{N} \log \mathbb{E} \mathbf{E} \left(e^{\sum_{n=1}^N (\beta \omega_n + h) \delta_n} \delta_N \right) = F_N(0, h + \beta^2/2). \quad (2.5)$$

(If disorder is not Gaussian, $\beta^2/2$ is replaced by $\log \mathbb{E} \exp(\beta \omega_1)$.) $F_N^a(\beta, h)$ is known as the (finite-volume) *annealed* free energy which, as (2.5) shows, coincides with the free energy of the homogeneous model ($\beta = 0$) for a shifted value of h . The limit free energy $F(\beta, h)$ would not change (see, e.g., [11, Remark 1.1]) if the factor δ_N were omitted in (2.3), i.e., if the boundary condition $\{N \in \tau\}$ were replaced by a free boundary condition at N . However, in that case exact superadditivity, and (2.4), would not hold.

Another well-established fact is that $F(\beta, h) \geq 0$ (cf. for instance [11]), which allows the definition of the critical point, for a given $\beta \geq 0$, as $h_c(\beta) := \sup\{h \in \mathbb{R} : F(\beta, h) = 0\}$. Note that Eq. (2.5) implies $h_c(\beta) \geq h_c^a(\beta) := \sup\{h \in \mathbb{R} : F^a(\beta, h) = 0\} = h_c(0) - \beta^2/2$. For obvious reasons, $h_c^a(\beta)$ is referred to as the *annealed critical point*. Concerning upper bounds for $h_c(\beta)$, already before [2] it was known (see [3] and [9, Theorem 5.2]) that $h_c(\beta) < h_c(0)$ for every $\beta > 0$. To make a link with the discussion in the introduction, note that the contact fraction $\ell(\beta, h)$ is just $\partial_h F(\beta, h)$.

With the exception of Theorem 2.6, we will consider from now on only the values $0 < \alpha < 1$, as in [2], in which case τ is null-recurrent under \mathbf{P} . For the homogeneous system it is known [9, Theorem 2.1] that $F(0, h) = 0$ if $h \leq 0$, while for $h > 0$

$$F(0, h) = h^{1/\alpha} \tilde{L}(1/h). \quad (2.6)$$

$\tilde{L}(\cdot)$ is a slowly varying function satisfying

$$\tilde{L}(1/h) = \left(\frac{\alpha}{\Gamma(1-\alpha)} \right)^{1/\alpha} h^{-1/\alpha} R_\alpha(h), \quad (2.7)$$

and $R_\alpha(\cdot)$ is asymptotically equivalent to the inverse of the map $x \mapsto x^\alpha L(1/x)$. The fact that $\tilde{L}(\cdot)$ is slowly varying follows from [4, Theorem 1.5.12]. In particular, notice that $h_c(0) = 0$, so that $h_c^a(\beta) = -\beta^2/2$.

We want to prove first of all that, if $0 < \alpha < 1/2$ and β is sufficiently small (i.e., if disorder is sufficiently weak), $h_c(\beta) = h_c^a(\beta)$. Keeping in mind that $F^a(\beta, h_c^a(\beta) + \Delta) = F(0, \Delta)$, this is an immediate consequence of

Theorem 2.1. *Assume that either $0 < \alpha < 1/2$ or that $\alpha = 1/2$ and $\sum_{n \in \mathbb{N}} n^{-1} L(n)^{-2} < \infty$. Then, for every $\epsilon > 0$ there exist $\beta_0(\epsilon) > 0$ and $\Delta_0(\epsilon) > 0$ such that, for every $\beta \leq \beta_0(\epsilon)$ and $0 < \Delta < \Delta_0(\epsilon)$, one has*

$$(1 - \epsilon)F(0, \Delta) \leq F(\beta, h_c^a(\beta) + \Delta) \leq F(0, \Delta). \quad (2.8)$$

In view of [11, Theorem 2.1], the same cannot hold for $\alpha > 1/2$. However, one has:

Theorem 2.2. *Assume that $1/2 < \alpha < 1$. There exists a slowly varying function $\check{L}(\cdot)$ and, for every $\epsilon > 0$, constants $a_1(\epsilon) < \infty$ and $\Delta_0(\epsilon) > 0$ such that, if*

$$a_1(\epsilon) \beta^{2\alpha/(2\alpha-1)} \check{L}(1/\beta) \leq \Delta \leq \Delta_0(\epsilon), \quad (2.9)$$

the inequalities (2.8) hold.

As already pointed out in [2], since $2\alpha/(2\alpha-1) > 2$ Theorem 2.2 shows in particular that

$$\lim_{\beta \searrow 0} \frac{h_c(\beta)}{h_c^a(\beta)} = 1. \quad (2.10)$$

On the other hand, it is unknown whether there exist non-zero values of β such that the equality $h_c(\beta) = h_c^a(\beta)$ holds, for $1/2 < \alpha < 1$.

Remark 2.3. The lower bound we obtain for $F(\beta, h)$ (and, as a consequence, for $h_c^a(\beta) - h_c(\beta)$) in Theorem 2.2 differs from the analogous one of [2, Theorem 3] only in the form of the slowly varying function $\check{L}(\cdot)$ (an explicit choice of $\check{L}(\cdot)$ can be extracted from Eq. (3.28) below). In general, our $\check{L}(\cdot)$ is larger due to the logarithmic denominator in (3.28). However, the important point is that the exponent of β in (2.9) agrees with that in the analogous condition of [2, Theorem 3]. Indeed, with the conventions of [2] (which amount to replacing everywhere h by βh and α by $c-1$), the exponent $2\alpha/(2\alpha-1) = 1 + 1/(2\alpha-1)$ would be instead $1/(2\alpha-1) = 1/(2c-3)$, as in [2, Theorem 3].

Finally, for the “marginal case” we have

Theorem 2.4. Assume that $\alpha = 1/2$ and $\sum_{n \in \mathbb{N}} n^{-1} L(n)^{-2} = \infty$. Let $\ell(\cdot)$ be the slowly varying function (diverging at infinity) defined by

$$\sum_{n=1}^N \frac{1}{nL(n)^2} \stackrel{N \rightarrow \infty}{\sim} \ell(N). \quad (2.11)$$

For every $\epsilon > 0$ there exist constants $a_2(\epsilon) < \infty$ and $\Delta_0(\epsilon) > 0$ such that, if $0 < \Delta \leq \Delta_0(\epsilon)$ and if the condition

$$\frac{1}{\beta^2} \geq a_2(\epsilon) \ell \left(\frac{a_2(\epsilon) |\log F(0, \Delta)|}{F(0, \Delta)} \right) \quad (2.12)$$

is verified, then Eq. (2.8) holds.

Remark 2.5. Note that, thanks to Theorem 2.4 and the property of slow variation of $\ell(\cdot)$, the difference $h_c(\beta) - h_c^a(\beta)$ vanishes faster than any power of β , for $\beta \searrow 0$. Again, it is unknown whether $h_c(\beta) = h_c^a(\beta)$ for some $\beta > 0$.

In general, our condition (2.12) is different from the one in the analogous Theorem 4 of [2], due to the presence of the factor $|\log F(0, \Delta)|$ in the argument of $\ell(\cdot)$. However, for many “reasonable” and physically interesting choices of $L(\cdot)$ in (2.1), the two results are equivalent. In particular, if \mathbf{P} is the law of the returns to zero of the SRW $\{S_n\}_{n \geq 0}$ in one dimension, i.e. $\tau = \{n \geq 0 : S_{2n} = 0\}$, in which case $L(\cdot)$ and $\tilde{L}(\cdot)$ are asymptotically constant and $\ell(N) \sim a_3 \log N$, one sees easily that (2.12) is verified as soon as

$$\Delta \geq a_4(\epsilon) e^{-\frac{a_5(\epsilon)}{\beta^2}}, \quad (2.13)$$

which is the same condition which was found in [2]. Another case where Theorem 2.4 and [2, Theorem 4] are equivalent is when $L(n) \sim a_6(\log n)^{(1-\gamma)/2}$ for $\gamma > 0$, in which case $\ell(N) \sim a_7(\log N)^\gamma$.

While we focused up to now on free energy lower bounds, one may wonder whether it is possible to improve the Jensen upper bound (2.5). For $\alpha > 1/2$ it follows from [11] that $F(\beta, h) < F^a(\beta, h)$ as soon as β is positive and $h - h_c^a(\beta)$ is positive and small. We conclude this section with a theorem which generalizes this result to arbitrary α and $h > h_c^a(\beta)$, and which justifies (as an upper bound) a small- β expansion worked out in [9, Section 5.5].

Theorem 2.6. For every $\beta > 0$, $\alpha \geq 0$ and $\Delta > 0$

$$F(\beta, h_c^a(\beta) + \Delta) \leq \inf_{0 \leq q \leq \Delta/\beta^2} \left(\frac{\beta^2 q^2}{2} + F(0, \Delta - \beta^2 q) \right) < F(0, \Delta). \quad (2.14)$$

As a consequence, for $0 \leq \alpha < 1/2$ there exist $\beta_0 > 0$ and $\Delta_0 > 0$ such that

$$F(\beta, h_c^a(\beta) + \Delta) \leq F(0, \Delta) - \frac{\beta^2}{2} (\partial_\Delta F(0, \Delta))^2 (1 + O(\beta^2)) \quad (2.15)$$

for $\beta \leq \beta_0$ and $\Delta \leq \Delta_0$, where $O(\beta^2)$ is independent of Δ .

The first inequality in (2.14) is somewhat reminiscent of the “replica-symmetric” free energy upper bound [13] for the Sherrington-Kirkpatrick model.

The reader who wonders why we stopped at order β^2 in the “expansion” (2.15) should look at Remark 3.1 below. Note that, in view of Eqs. (2.6) and (3.47), $(\partial_\Delta F(0, \Delta))^2 \ll F(0, \Delta)$ if Δ is small and $\alpha < 1/2$. Observe also that, for $\alpha > 1/2$ and β, Δ small, taking the infimum in (2.14) gives nothing substantially better than just choosing $q = \Delta/\beta^2$,

from which $F(\beta, h_c^a(\beta) + \Delta) \leq \Delta^2/(2\beta^2)$; essentially the same bound (with an extra factor $(1 + \alpha)$ in the right-hand side) is however already implied by [11, Theorem 2.1] (see also [9, Remark 5.7]).

Remark 2.7. As a general remark, we emphasize that the assumption of recurrence for τ , i.e., $\sum_{n \in \mathbb{N}} K(n) = 1$ is by no means a restriction. Indeed, as has been observed several times in the literature (including [11] and [2]), if $\Sigma_K := \sum_{n \in \mathbb{N}} K(n) < 1$ one can define $\tilde{K}(n) := K(n)/\Sigma_K$, and of course the renewal τ with law $\tilde{\mathbf{P}}(\tau_1 = n) = \tilde{K}(n)$ is recurrent. Then, it is immediate to realize from definition (2.3) that

$$F(\beta, h) = \tilde{F}(\beta, h + \log \Sigma_K), \quad (2.16)$$

\tilde{F} being the free energy of the model defined as in (2.3) but with \mathbf{P} replaced by $\tilde{\mathbf{P}}$. In particular, $h_c^a(\beta) = -\log \Sigma_K - \beta^2/2$. Theorems 2.1-2.6 are therefore transferred with obvious changes to this situation.

This observation allows to apply the results, for instance, to the case where we consider the SRW $\{S_n\}_{n \geq 0}$ in \mathbb{Z}^3 , and we let \mathbf{P} be the law of $\tau := \{n \geq 0 : S_{2n} = 0\}$, i.e., the law of its returns to the origin. In this case, assumption (2.1) holds with $\alpha = 1/2$, $L(\cdot)$ asymptotically constant and, due to transience, $\Sigma_K < 1$. The same is true if $\{S_n\}_{n \geq 0}$ is the SRW on \mathbb{Z} , conditioned to be non-negative.

3. PROOFS

Proof of Theorem 2.1. In view of Eq. (2.5), we have to prove only the first inequality in (2.8). This is based on an adaptation of the *quadratic replica coupling method* of [15], plus ideas suggested by [2]. Let $\Delta > 0$ and start from the identity

$$F(\beta, -\beta^2/2 + \Delta) = F(0, \Delta) + \lim_{N \rightarrow \infty} R_{N, \Delta}(\beta) \quad (3.1)$$

where

$$R_{N, \Delta}(\beta) := \frac{1}{N} \mathbb{E} \log \left\langle e^{\sum_{n=1}^N (\beta \omega_n - \beta^2/2) \delta_n} \right\rangle_{N, \Delta},$$

and, for a \mathbf{P} -measurable function $f(\tau)$,

$$\langle f \rangle_{N, \Delta} := \frac{\mathbf{E} \left(e^{\Delta \sum_{n=1}^N \delta_n} f \delta_N \right)}{\mathbf{E} \left(e^{\Delta \sum_{n=1}^N \delta_n} \delta_N \right)}. \quad (3.2)$$

Via the Gaussian integration by parts formula

$$\mathbb{E}(\omega f(\omega)) = \mathbb{E}f'(\omega), \quad (3.3)$$

valid (if ω is a standard Gaussian random variable $\mathcal{N}(0, 1)$) for every differentiable function $f(\cdot)$ such that $\lim_{|x| \rightarrow \infty} \exp(-x^2/2) f(x) = 0$, one finds for $0 < t < 1$:

$$\frac{d}{dt} R_{N, \Delta}(\sqrt{t}\beta) = -\frac{\beta^2}{2N} \sum_{m=1}^N \mathbb{E} \left\{ \left(\frac{\left\langle \delta_m e^{\sum_{n=1}^N (\beta \sqrt{t} \omega_n - t\beta^2/2) \delta_n} \right\rangle_{N, \Delta}}{\left\langle e^{\sum_{n=1}^N (\beta \sqrt{t} \omega_n - t\beta^2/2) \delta_n} \right\rangle_{N, \Delta}} \right)^2 \right\}. \quad (3.4)$$

Define also, for $\lambda \geq 0$,

$$\begin{aligned} \psi_{N, \Delta}(t, \lambda, \beta) &:= \frac{1}{2N} \mathbb{E} \log \left\langle e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N, \Delta}^{\otimes 2} \\ &:= \frac{1}{2N} \mathbb{E} \log \left\langle e^{\sum_{n=1}^N (\beta \sqrt{t} \omega_n - t\beta^2/2) (\delta_n^{(1)} + \delta_n^{(2)}) + \lambda \beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)}} \right\rangle_{N, \Delta}^{\otimes 2} \end{aligned} \quad (3.5)$$

where the product measure $\langle \cdot \rangle_{N,\Delta}^{\otimes 2}$ acts on the pair $(\tau^{(1)}, \tau^{(2)})$, while $\delta_n^{(i)} := \mathbf{1}_{n \in \tau^{(i)}}$. Note that $\psi_{N,\Delta}(t, \lambda, \beta)$ actually depends on (t, λ, β) only through the two combinations $\beta^2 t$ and $\beta^2 \lambda$. We add also that the introduction of the parameter t , which could in principle be avoided, allows for more natural expressions in the formulas which follow. One has immediately

$$\psi_{N,\Delta}(0, \lambda, \beta) = \frac{1}{2N} \log \left\langle e^{\lambda \beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)}} \right\rangle_{N,\Delta}^{\otimes 2} \quad (3.6)$$

and

$$\psi_{N,\Delta}(t, 0, \beta) = R_{N,\Delta}(\sqrt{t}\beta). \quad (3.7)$$

Again via integration by parts,

$$\begin{aligned} \frac{d}{dt} \psi_{N,\Delta}(t, \lambda, \beta) &= \frac{\beta^2}{2N} \sum_{m=1}^N \mathbb{E} \frac{\left\langle \delta_m^{(1)} \delta_m^{(2)} e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta}^{\otimes 2}}{\left\langle e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta}^{\otimes 2}} \\ &\quad - \frac{\beta^2}{4N} \sum_{m=1}^N \mathbb{E} \left\{ \left(\frac{\left\langle (\delta_m^{(1)} + \delta_m^{(2)}) e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta}^{\otimes 2}}{\left\langle e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta}^{\otimes 2}} \right)^2 \right\} \\ &\leq \frac{\beta^2}{2N} \mathbb{E} \sum_{m=1}^N \frac{\left\langle \delta_m^{(1)} \delta_m^{(2)} e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta}^{\otimes 2}}{\left\langle e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta}^{\otimes 2}} = \frac{d}{d\lambda} \psi_{N,\Delta}(t, \lambda, \beta), \end{aligned} \quad (3.8)$$

so that, for every $0 \leq t \leq 1$ and λ ,

$$\psi_{N,\Delta}(t, \lambda, \beta) \leq \psi_{N,\Delta}(0, \lambda + t, \beta). \quad (3.9)$$

Going back to Eq. (3.4), using convexity and monotonicity of $\psi_{N,\Delta}(t, \lambda, \beta)$ with respect to λ and (3.7), one finds

$$\begin{aligned} \frac{d}{dt} \left(-R_{N,\Delta}(\sqrt{t}\beta) \right) &= \frac{d}{d\lambda} \psi_{N,\Delta}(t, \lambda, \beta)|_{\lambda=0} \leq \frac{\psi_{N,\Delta}(t, 2-t, \beta) - R_{N,\Delta}(\sqrt{t}\beta)}{2-t} \\ &\leq \psi_{N,\Delta}(0, 2, \beta) - R_{N,\Delta}(\sqrt{t}\beta), \end{aligned} \quad (3.10)$$

where in the last inequality we used (3.9) and the fact that $2-t \geq 1$. Integrating this differential inequality between 0 and 1 and observing that $R_{N,\Delta}(0) = 0$, one has

$$0 \leq -R_{N,\Delta}(\beta) \leq (e-1)\psi_{N,\Delta}(0, 2, \beta). \quad (3.11)$$

Now we estimate

$$\begin{aligned} \psi_{N,\Delta}(0, 2, \beta) &= -F_N(0, \Delta) + \frac{1}{2N} \log \mathbf{E}^{\otimes 2} \left(e^{2\beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)} + \Delta \sum_{n=1}^N (\delta_n^{(1)} + \delta_n^{(2)})} \delta_N^{(1)} \delta_N^{(2)} \right) \\ &\leq -F_N(0, \Delta) + \frac{F_N(0, q\Delta)}{q} + \frac{1}{2Np} \log \mathbf{E}^{\otimes 2} \left(e^{2p\beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)}} \right) \end{aligned} \quad (3.12)$$

where we used Hölder's inequality and p, q (satisfying $1/p + 1/q = 1$) are to be determined. One finds then

$$\limsup_{N \rightarrow \infty} \psi_{N, \Delta}(0, 2, \beta) \leq \limsup_{N \rightarrow \infty} \frac{1}{2Np} \log \mathbf{E}^{\otimes 2} \left(e^{2p\beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)}} \right) + F(0, \Delta) \left(\frac{1}{q} \frac{F(0, q\Delta)}{F(0, \Delta)} - 1 \right). \quad (3.13)$$

But we know from (2.6) and the property (2.2) of slow variation that, for every $q > 0$,

$$\lim_{\Delta \searrow 0} \frac{F(0, q\Delta)}{F(0, \Delta)} = q^{1/\alpha}. \quad (3.14)$$

Therefore, choosing $q = q(\epsilon)$ sufficiently close to (but not equal to) 1 and $\Delta_0(\epsilon) > 0$ sufficiently small one has, uniformly on $\beta \geq 0$ and on $0 < \Delta \leq \Delta_0(\epsilon)$,

$$\limsup_{N \rightarrow \infty} \psi_{N, \Delta}(0, 2, \beta) \leq \frac{\epsilon}{e-1} F(0, \Delta) + \limsup_{N \rightarrow \infty} \frac{1}{2Np(\epsilon)} \log \mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)}} \right) \quad (3.15)$$

Of course, $p(\epsilon) = q(\epsilon)/(q(\epsilon) - 1) < \infty$ as long as $\epsilon > 0$. Finally, we observe that under the assumptions of the theorem, the renewal $\tau^{(1)} \cap \tau^{(2)}$ is transient under the law $\mathbf{P}^{\otimes 2}$. Indeed, if $0 < \alpha < 1/2$ or if $\alpha = 1/2$ and $\sum_{n \in \mathbb{N}} n^{-1} L(n)^{-2} < \infty$ one has

$$\mathbf{E}^{\otimes 2} \left(\sum_{n \geq 1} \mathbf{1}_{n \in \tau^{(1)} \cap \tau^{(2)}} \right) = \sum_{n \geq 1} \mathbf{P}(n \in \tau)^2 < \infty \quad (3.16)$$

since, as proven in [7],

$$\mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \frac{C_\alpha}{L(n)n^{1-\alpha}} := \frac{\alpha \sin(\pi\alpha)}{\pi} \frac{1}{L(n)n^{1-\alpha}}. \quad (3.17)$$

Actually, Eq. (3.17) holds more generally for $0 < \alpha < 1$.

Therefore, there exists $\beta_1 > 0$ such that

$$\sup_N \mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)}} \right) < \infty \quad (3.18)$$

for every $\beta^2 p(\epsilon) \leq \beta_1^2$. Together with (3.15) and (3.1), this implies

$$F(\beta, -\beta^2/2 + \Delta) \geq (1 - \epsilon)F(0, \Delta) \quad (3.19)$$

as soon as $\beta^2 \leq \beta_0^2(\epsilon) := \beta_1^2/p(\epsilon)$. \square

Proof of Theorem 2.2. In what follows we assume that Δ is sufficiently small so that $F(0, \Delta) < 1$. Let $N = N(\Delta) := c |\log F(0, \Delta)|/F(0, \Delta)$ with $c > 0$. By Eq. (2.4) we have, in analogy with (3.1),

$$F(\beta, -\beta^2/2 + \Delta) \geq F_{N(\Delta)}(0, \Delta) + R_{N(\Delta), \Delta}(\beta). \quad (3.20)$$

As follows from Proposition 2.7 of [10], there exists $a_8 \in (0, \infty)$ (depending only on the law $K(\cdot)$ of the renewal) such that

$$F_N(0, \Delta) \geq F(0, \Delta) - a_8 \frac{\log N}{N} \quad (3.21)$$

for every N . Choosing $c = c(\epsilon)$ large enough, Eq. (3.21) implies that

$$F_{N(\Delta)}(0, \Delta) \geq (1 - \epsilon)F(0, \Delta). \quad (3.22)$$

As for $R_{N(\Delta),\Delta}(\beta)$, we have from (3.11) and (3.12)

$$(1-e)^{-1}R_{N(\Delta),\Delta}(\beta) \leq F(0, \Delta) \left(\frac{1}{q} \frac{F(0, q\Delta)}{F(0, \Delta)} - 1 \right) + \epsilon F(0, \Delta) + \frac{1}{2N(\Delta)p} \log \mathbf{E}^{\otimes 2} \left(e^{2p\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right), \quad (3.23)$$

where we used Eqs. (3.22) and (2.4) to bound $(1/q)F_{N(\Delta)}(0, q\Delta) - F_{N(\Delta)}(0, \Delta)$ from above. Choosing again $q = q(\epsilon)$ we obtain, for $\Delta \leq \Delta_0(\epsilon)$,

$$(1-e)^{-1}R_{N(\Delta),\Delta}(\beta) \leq 2\epsilon F(0, \Delta) + \frac{1}{2N(\Delta)p(\epsilon)} \log \mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right). \quad (3.24)$$

Now observe that, if $1/2 < \alpha < 1$, there exists $a_9 = a_9(\alpha) \in (0, \infty)$ such that for every integers N and k

$$\mathbf{P}^{\otimes 2} \left(\sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)} \geq k \right) \leq \left(1 - a_9 \frac{L(N)^2}{N^{2\alpha-1}} \right)^k. \quad (3.25)$$

This geometric bound is proven in [2, Lemma 3], but in Subsection 3.1 we give another simple proof. Thanks to (3.25) we have

$$\mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \left(1 - e^{2\beta^2 p(\epsilon)} \left(1 - a_9 \frac{L(N(\Delta))^2}{N(\Delta)^{2\alpha-1}} \right) \right)^{-1}, \quad (3.26)$$

whenever the right-hand side is positive, and this is of course the case under the stronger requirement

$$e^{2\beta^2 p(\epsilon)} \left(1 - a_9 \frac{L(N(\Delta))^2}{N(\Delta)^{2\alpha-1}} \right) \leq \left(1 - \frac{a_9}{2} \frac{L(N(\Delta))^2}{N(\Delta)^{2\alpha-1}} \right). \quad (3.27)$$

At this point, using the definition of $N(\Delta)$, it is not difficult to see that there exists a positive constant $a_{10}(\epsilon)$ such that (3.27) holds if

$$\begin{aligned} \beta^2 p(\epsilon) &\leq a_{10}(\epsilon) \Delta^{(2\alpha-1)/\alpha} \hat{L}(1/\Delta) \\ &:= a_{10}(\epsilon) \Delta^{(2\alpha-1)/\alpha} \left[\frac{\tilde{L}(1/\Delta)}{|\log F(0, \Delta)|} \right]^{2\alpha-1} \left(L \left(\frac{|\log F(0, \Delta)|}{F(0, \Delta)} \right) \right)^2. \end{aligned} \quad (3.28)$$

The fact that $\hat{L}(\cdot)$ is slowly varying follows from [4, Proposition 1.5.7] and Eq. (2.6). For instance, if $L(\cdot)$ is asymptotically constant one has $\hat{L}(x) \sim a_{11} |\log x|^{1-2\alpha}$. Condition (3.28) is equivalent to the first inequality in (2.9), for suitably chosen $a_1(\epsilon)$ and $\check{L}(\cdot)$. As a consequence,

$$\frac{1}{2N(\Delta)p(\epsilon)} \log \mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \frac{F(0, \Delta)}{2c(\epsilon)p(\epsilon)|\log F(0, \Delta)|} \log \left(\frac{2N(\Delta)^{2\alpha-1}}{a_9 L(N(\Delta))^2} \right) \quad (3.29)$$

Recalling Eq. (2.6) one sees that, if $c(\epsilon)$ is chosen large enough,

$$\frac{1}{2N(\Delta)p(\epsilon)} \log \mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \epsilon F(0, \Delta). \quad (3.30)$$

Together with Eqs. (3.20), (3.22) and (3.24), this concludes the proof of the theorem. \square

Proof of Theorem 2.4. The proof is almost identical to that of Theorem 2.2 and up to Eq. (3.24) no changes are needed. The estimate (3.25) is then replaced by

$$\mathbf{P}^{\otimes 2} \left(\sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)} \geq k \right) \leq \left(1 - \frac{a_{12}}{\ell(N)} \right)^k. \quad (3.31)$$

for every N , for some $a_{12} > 0$ (see [2, Lemma 3], or the alternative argument given in Subsection 3.1). In analogy with Eq. (3.26) one obtains then

$$\mathbf{E}^{\otimes 2} \left(e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \left(1 - e^{2\beta^2 p(\epsilon)} \left(1 - \frac{a_{12}}{\ell(N(\Delta))} \right) \right)^{-1} \quad (3.32)$$

whenever the right-hand side is positive. Choosing $a_2(\epsilon)$ large enough one sees that if condition (2.12) is fulfilled then

$$e^{2\beta^2 p(\epsilon)} \left(1 - \frac{a_{12}}{\ell(N(\Delta))} \right) \leq \left(1 - \frac{a_{12}}{2\ell(N(\Delta))} \right) \quad (3.33)$$

and, in analogy with (3.29),

$$\frac{1}{2N(\Delta)p(\epsilon)} \log \mathbf{E}^{\otimes 2} \left(e^{2(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \frac{F(0, \Delta)}{2c(\epsilon)p(\epsilon)|\log F(0, \Delta)|} \log \left(\frac{2\ell(N(\Delta))}{a_{12}} \right) \quad (3.34)$$

From this estimate, for $c(\epsilon)$ sufficiently large one obtains again (3.30) and as a consequence the statement of Theorem 2.4. \square

3.1. Proof of (3.25) and (3.31). For what concerns (3.25), start from the obvious bound

$$\mathbf{P}^{\otimes 2} \left(\sum_{n=1}^N \delta_n^{(1)} \delta_n^{(2)} \geq k \right) \leq \left(1 - \mathbf{P}^{\otimes 2}(\inf\{n > 0 : n \in \tau^{(1)} \cap \tau^{(2)}\} > N) \right)^k. \quad (3.35)$$

Next note that, by Eq. (3.17),

$$u_n := \mathbf{P}^{\otimes 2}(n \in \tau^{(1)} \cap \tau^{(2)}) \stackrel{n \rightarrow \infty}{\sim} \frac{C_\alpha^2}{L(n)^2 n^{2(1-\alpha)}} \quad (3.36)$$

and that u_n satisfies the renewal equation

$$u_n = \delta_{n,0} + \sum_{k=0}^{n-1} u_k Q(n-k) \quad (3.37)$$

where $Q(k) := \mathbf{P}^{\otimes 2}(\inf\{n > 0 : n \in \tau^{(1)} \cap \tau^{(2)}\} = k)$ is the probability we need to estimate in (3.35). $Q(\cdot)$ is a probability on \mathbb{N} since the renewal $\tau^{(1)} \cap \tau^{(2)}$ is recurrent for $1/2 < \alpha < 1$, as can be seen from the fact that, due to Eq. (3.17), the expectation in (3.16) diverges in this case. After a Laplace transform, one finds for $s > 0$

$$\hat{Q}(s) := \sum_{n \geq 0} e^{-ns} Q(n) = 1 - \frac{1}{\hat{u}(s)} \quad (3.38)$$

and, by [4, Theorem 1.7.1] and the asymptotic behavior (3.36), one finds

$$\hat{u}(s) \stackrel{s \rightarrow 0^+}{\sim} \frac{C_\alpha^2 \Gamma(2\alpha)}{2\alpha - 1} \frac{1}{s^{2\alpha-1} (L(1/s))^2}. \quad (3.39)$$

Note that $0 < 2\alpha - 1 < 1$. By the classical Tauberian theorem (in particular, [4, Corollary 8.1.7] is enough in this case), one obtains then

$$\sum_{n \geq N} Q_n \stackrel{N \rightarrow \infty}{\sim} \frac{2\alpha - 1}{C_\alpha^2 \Gamma(2\alpha) \Gamma(2(1 - \alpha))} \frac{L(N)^2}{N^{2\alpha - 1}} \quad (3.40)$$

which, together with (3.35), completes the proof of (3.25).

We turn now to the proof of (3.31). From Eq. (3.36) with $\alpha = 1/2$ and [4, Theorem 1.7.1], one finds, in analogy with (3.39),

$$\hat{u}(s) \stackrel{s \rightarrow 0^+}{\sim} C_{1/2}^2 \ell(1/s). \quad (3.41)$$

Then, Eq. (3.38) and [4, Corollary 8.1.7] imply

$$\sum_{n \geq N} Q_n \stackrel{N \rightarrow \infty}{\sim} \frac{1}{C_{1/2}^2 \ell(N)} \quad (3.42)$$

and therefore Eq. (3.31). \square

3.2. Proof of Theorem 2.6. Start again from (3.1) and define, for $q \in \mathbb{R}$,

$$\phi_{N,\Delta}(t, \beta) := \frac{1}{N} \mathbb{E} \log \left\langle e^{\sum_{n=1}^N [\beta \sqrt{t} \omega_n - t\beta^2/2 + \beta^2 q(t-1)] \delta_n} \right\rangle_{N,\Delta} \quad (3.43)$$

so that

$$\phi_{N,\Delta}(0, \beta) = F_N(0, \Delta - \beta^2 q) - F_N(0, \Delta) \quad (3.44)$$

and $\phi_{N,\Delta}(1, \beta) = R_{N,\Delta}(\beta)$. In analogy with Eq. (3.4) one has

$$\begin{aligned} \frac{d}{dt} \phi_{N,\Delta}(t, \beta) &= -\frac{\beta^2}{2N} \sum_{m=1}^N \mathbb{E} \left\{ \left(\frac{\left\langle \delta_m e^{\sum_{n=1}^N [\beta \sqrt{t} \omega_n - t\beta^2/2 + \beta^2 q(t-1)] \delta_n} \right\rangle_{N,\Delta}}{\left\langle e^{\sum_{n=1}^N [\beta \sqrt{t} \omega_n - t\beta^2/2 + \beta^2 q(t-1)] \delta_n} \right\rangle_{N,\Delta}} - q \right)^2 \right\} + \frac{\beta^2 q^2}{2} \\ &\leq \frac{\beta^2 q^2}{2}, \end{aligned} \quad (3.45)$$

from which statement (2.14) follows after an integration on t (it is clear that taking the infimum over $q \in \mathbb{R}$ or over $0 \leq q \leq \Delta/\beta^2$ gives the same result.) The strict inequality in (2.14) holds since the quantity to be minimized in (2.14) has negative derivative at $q = 0$.

To prove (2.15) recall that $F(0, \Delta)$ satisfies for $\Delta > 0$ the identity [11, Appendix A]

$$\sum_{n \in \mathbb{N}} e^{-F(0, \Delta)n} K(n) = e^{-\Delta}, \quad (3.46)$$

(so that, in particular, $F(0, \Delta)$ is real analytic for $\Delta > 0$). An application of [4, Theorem 1.7.1] gives therefore, for $\alpha < 1/2$,

$$\partial_\Delta F(0, \Delta) = \Delta^{(1-\alpha)/\alpha} L^{(1)}(1/\Delta), \quad \partial_\Delta^2 F(0, \Delta) = \Delta^{(1-2\alpha)/\alpha} L^{(2)}(1/\Delta), \quad (3.47)$$

where the slowly varying functions $L^{(i)}(\cdot)$ can be expressed through $L(\cdot)$ (cf., for instance, [9, Section 2.4] for the first equality). For $\alpha = 0$, (3.47) is understood to mean that the two derivatives vanish faster than any power of Δ . This shows that $\partial_\Delta^2 F(0, \Delta)$ is bounded above by a constant for, say, $\Delta \leq 1$ if $\alpha < 1/2$. Then, choosing $q = \partial_\Delta F(0, \Delta)$ in (2.14) (which is the minimizer of $\beta^2 q^2/2 + F(0, \Delta - \beta^2 q)$ at lowest order in β) yields (2.15). It is important to note that, thanks to the first equality in (3.47) and the assumption $\alpha < 1/2$, this choice is compatible with the constraint $q \leq \Delta/\beta^2$, for Δ and β sufficiently small. \square

Remark 3.1. The reason why we stopped at order β^2 in (2.15) is that at next order the error term $O(\beta^6)$ involves $\partial_\Delta^3 F(0, \Delta)$, which diverges for $\Delta \searrow 0$ if $\alpha > 1/3$. In analogy with (2.15), one can however prove that, if $\alpha < 1/k$ with $2 < k \in \mathbb{N}$, the expansion (2.15) can be pushed to order $\beta^{2(k-1)}$ with a uniform control in Δ of the error term $O(\beta^{2k})$. We do not detail this point, the computations involved being straightforward.

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